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Research Article

Mathematical Modeling of Blood Flow through an Artery with Multiple Constrictions

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ABSTRACT: *Mathematical model is a method of simulating real-life situations with mathematical equations to predict their future behavior. Blood vessels are the channels or conduit through which blood is distributed to body tissues .This study is concerned with mathematical models of blood flow behavior through an artery having multiple constrictions. The stenoses is assumed to be mild and using the no slip boundary condition, the equations governing the flow of the proposed model are solved and closed form expressions for the haemodynamic factors like wall shear stress, volumetric flow rate, pressure and velocity are analyzed theoretically.*

KEYWORDS: *blood flow, constriction, flow resistance, pressure, stenosis, wall shear stress*

I. INTRODUCTION

One of the oldest branches of Applied Mathematics is fluid dynamics, which is one of the most important part of the inter-displinary activities concerning biological, engineering and technological development. Researchers in the field of Applied Mathematics who are interested in biological and medical problems are concerned with the formulation of models/techniques for dealing with the complex situations in real life sciences. Once a model is formulated, its consequences can be deduced by using mathematical techniques and the results can be compared with observations. The discrepancies between theoretical conclusions and observations suggest further improvements in the model. Classical mathematical techniques such as the solution of ordinary and partial differential equations, integral equations, probabilistic and statistical techniques are used in these model. Most often' the differential and integral equations arising in mathematical biosciences can be solved only with the help of computers. The study of blood flow in the arteries is considered as bio-fluid dynamics.

According to WHO report 2012, cardiovascular diseasesare responsible for over17.5 million deaths per year and are the leading causes of deathin the world.Of these deaths, an estimated 6.7 million deaths were due to stroke. Stroke occurs when there is blockage or bleeding in the blood vessel that prevents blood from flowing to the brain. Flow in artery is the most common fluid dynamic phenomenon in biology. Fluids provide an indispensable medium for convection and diffusion of the numerous chemical and biochemical products required for the maintenance of biological functions, and arteries provide an efficient medium for the containment and transport of these fluids. Blood flow in arteries is dominated by unsteady flow phenomena. The cardiovascular system is an internal flow loop with multiple branches in which a complex liquid circulates. Cession or reduction in blood flow due to the presence of stenoses in the arteries is the single largest cause of death in most nations (Thiago, Claudia and Edson 2010). Arteries are living organs that can adapt toand change with unsteady hemodynamic conditions that requires an abnormal biological response.

One of the hemodynamic conditions is constriction of the artery that leads to abnormal narrowing of the arterial lumen. These constrictions usually occur in the large distributing arteries such as the coronary and carotid arteries. A common cause of this constriction is a chronic disease process called atherosclerosis. Atherosclerosis is a disease condition in which plaques build up inside the arteries. Plaque, include fatty streaks (lipid and foam cells); gelatinous plaques (collagen fibres around small lipid droplets); and fibrous plaques. These plaques cause abnormal narrowing of the arterial lumen called stenosis. The development of stenosis in an artery can have serious consequences and can disturb the normal functioning of the circulatory system. In particular, it maylead to increase in flow resistance, tissue damage leading to post stenotic dilation and increase danger of complete obstruction [1].

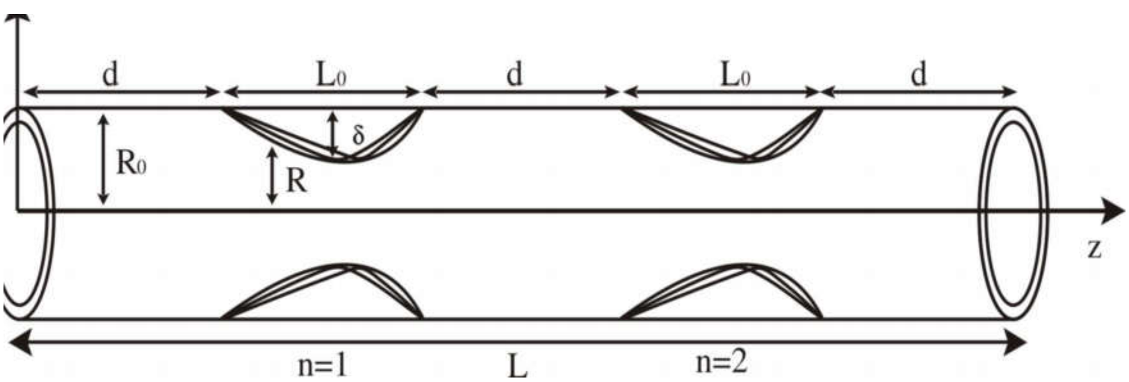
For many years, researchers have endeavored to model the flow of blood through stenosed arteries experimentally and theoretically. Sapna [2] worked on the mathematical model for the analysis of blood flow through diseased blood vessels under the influence of porous parameter. Varun and Varshney [3] studied pulsatile flow of blood through a stenosed artery with periodic body acceleration.Amit and shrivastar [4] studied analysis of MHD flow of blood through a multiple stenosed artery in the presence of Slip Velocity. Nidhi and Parimar [5]studiedmathematical model of blood flow through a tapered artery with mild stenosis and hematocrit.Hajeet et al [6] studied a non-Newtonian arterial blood flow model through multiple stenosis.Singh and Singh [7] studied the effect of hematocrit on wall shear stress for blood flow through tapered artery. Singh [8] also worked on the effect of shape parameter and length of stenosis on blood flow through improved generalized artery with multiple stenoses.Sapnah [9] investigated an innovative solution for the problem of blood flow through stenosed artery using generalized Bingham plastic fluid model. Shailesh et al [10] studied blood flow through a composite stenosis in an artery with permeable wall.Sahu et al [11] studied the arterial blood flow in stenosed vessel using non-newton couple stress fluid model. Chakravarty and Mandal [12] worked on mathematical modeling of blood flow through an overlapping arterial stenosis. Mukhopadhyay and Layek[13] studied the numerical modeling of a stenosed artery using mathematical model of variable shape.

Bearing this studies in mind, the aim of this present investigation is to study the effect of multiple stenoses on the flow ofblood through the artery. The blood which contains erythrocytes is represented by a Bingham fluid depicting thenon-Newtonian behavior of the blood. The derived analytical expressions are computed in orderto examine the variation of velocity profiles, the volumetric flow rate and the resistance to the blood flow.

The following assumptions were made for solving the model;
The liquid does not slip at the wall, flow is laminar (the liquid is moving parallel to the walls of the tube) ,the artery is cylindrical in shape (Most arteries of the systemic circulation are circularin cross-section) and it is assumed that stenosis is symmetrical about the axis but non- symmetrical with respect to radial coordinate.

II. MATHEMATICAL FORMULATION

We have investigated an artery having multiple stenoses. The flow of blood isassumed to be steady, laminar and fully-developed. Blood is taken as a Bingham fluid
The mathematical expression for the problem geometry is given by [9]



The mathematical expression of the geometry is given by:

$$\frac{R}{R_0} = \begin{cases} 1 - \frac{\delta_i}{2R_0}\left\{1 + \cos\frac{2\pi}{l_i}\left(z - \gamma_i - \frac{l_i}{2}\right)\right\} & for\ \gamma_i \le z \le \beta_i \\ 1, & otherwise \end{cases} \quad (1)$$

Where d_i is the maximum distance at i^{th} constricted segment, R_0 is the radius of normal artery, R is the radius of the constricted segment of the artery, $l_i$is the length of thei^{th} constricted segment , γ_i is the distance from the origin to the start of i^{th}constricted segment, $\beta_i$is the distance from the origin to the end of the i^{th} constriced segment.

Distance from the origin to the start of the i^{th}constricted segment is given by:

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$$\gamma_i = \left(\sum_{j=1}^{i}(d_j + l_j) - l_0\right) \qquad (2)$$

Distance from the origin to the end of i^{th} constricted segment is given by:

$$\beta_i = \left(\sum_{j=1}^{i} d_j + l_0\right) \quad (3)$$

The basic equation governing the flow is:

$$\frac{1}{r} = \frac{\partial(r\tau)}{\partial r} + \frac{\partial p}{\partial z} = 0 \quad (4)$$

Where w is the axial velocity, p is the pressure, τ_0 is the yield stress, μ is the fluid viscosity, ρ is the density, R_0 is the radius of the tube.
Taking blood to behave as a Bingham fluid, the constitutive equation for Bingham fluid is given by:

$$e = f(\tau) = -\frac{dw}{dr} = \begin{cases} \frac{1}{\mu}(\tau - \tau_0), & \tau \geq \tau_0 \\ 0, & \tau \leq \tau_0 \end{cases} \qquad (4b)$$

Boundary conditions are given as

$$\left.\begin{array}{lll} w = 0, & at & r = R \\ \frac{dw}{dr} = 0, & at & r = 0 \end{array}\right\} \qquad (5)$$

III. SOLUTION OF THE PROBLEM

3.1 Volumetric flux Q

The flux Q through the artery is given by

$$Q = \int_0^R 2\pi r w \, dr \quad (6)$$

Integrating (6) and using the boundary conditions in (5)

$$Q = \int_0^R r^2 \left(-\frac{dw}{dr}\right) dr \quad (7)$$

Applying (4) in (7) we have

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$Q = \pi \int_0^R r^2 f(\tau) dr$(8)

But wall shear stress $\tau = \tau_0$ when $r = R$is given by

$$\tau = -\frac{r}{2}\frac{dp}{dz} \text{and} \tau_R = -\frac{R}{2}\frac{dp}{dz} \qquad (9)$$

Where $\frac{dp}{dz}$ is the pressure gradient

From (4) and (6) we get

$$Q = \frac{\pi R^3}{\tau_R^3} \int_0^{\tau_R} \tau^2 f(\tau)\, d\tau \qquad (10)$$

Substituting (4) in (10)

$$Q = \frac{\pi R^3}{\mu \tau_R^3} \int_0^{\tau_R} \tau^2 (\tau - \tau_0) d\tau = \qquad (11)$$

Integrating (11), we have

$$Q = \frac{\pi R^3}{\mu} \left[\frac{\tau_R^4}{4} - \tau_0 \frac{\tau_R^3}{3}\right] \qquad (12)$$

3.2Wall shear stress

The wall shear stress is obtained

Where $\tau_R = \frac{4\mu Q}{\pi R^3} + \frac{\tau_0}{3}$ (13)

Wall shear stress in the absence of any constriction is given by

$$\tau_f = \frac{4\mu Q}{\pi R^3} \qquad (14)$$

Wall shear stress ratio$\bar{\tau}$ is given as

3.3 Pressure difference

$$\bar{\tau} = \frac{\tau_R}{\tau_f} = \frac{\pi R_0^{\,3}}{4\mu Q}\left[\frac{4\mu Q}{\pi R^3} + \frac{4\tau_0}{3}\right] \quad (15)$$

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$$= \frac{1}{\left(\frac{R}{R_0}\right)^3} + \frac{\pi R_0{}^3 \tau_0}{3\mu Q} \qquad (16)$$

Wall shear stress at the middle τ_δ of a constriction is given as

$$\tau_\delta = \frac{\pi R_0{}^3 \tau_0}{3\mu Q} - \frac{1}{\left(1-{\delta}/{R_0}\right)^3} \qquad (17)$$

The pressure difference Δp along the full length of the tube as follows

From (9), the pressure gradient is given as

$$\frac{dp}{dz} = \frac{2\tau_R}{R} \qquad (18)$$

From (13), the pressure gradient becomes

$$\frac{dp}{dz} = -\frac{2}{R}\left[\frac{\mu Q}{\pi R^3} + \frac{\tau_0}{3}\right] \qquad (19)$$

Integrating (19) with the condition that $p = p_1$ at $z = 0$, and $p = p_0$ at $z = l$

$$\int_{p_0}^{p_1} dp = \int_0^l \frac{\mu Q}{\pi R^4} - \frac{\tau_0}{3}\frac{1}{R} \qquad (20)$$

$$p_1 - p_0 = -\frac{2\tau_0}{3}\int_0^l \frac{1}{R}\,dz - \frac{\mu Q}{\pi}\int_0^l \frac{1}{R^4}\,dz \qquad (21)$$

3.4 Resistance to flow

Resistance of the flow λ is calculated as

$$\text{But } \lambda = \frac{p_1 - p_0}{Q} = -\frac{2\tau_0}{3QR_0}\int_0^l \left(\frac{R}{R_0}\right)^{-1} dz - \frac{\mu}{\pi R_0{}^4}\int_0^l \left(\frac{R}{R_0}\right)^{-4} dz \qquad (22)$$

$$\text{Let } h_1 = \frac{2\tau_0}{3QR_0}; \qquad h_2 = \frac{\mu}{\pi R_0{}^4} \qquad (23)$$

Therefore,

$$\lambda = -h_1\left[\int_0^{\gamma_i} dz + \sum_{i=1}^{n}\int_{\gamma_i}^{\beta_i}\left(\frac{R}{R_0}\right)^{-4} dz + \sum_{i=1}^{n-1}\int_{\beta_i}^{\gamma_{i+1}} dz + \int_{\beta_n}^{l} dz\right]$$

$$+h_2\left[\int_0^{\gamma_i} dz + \sum_{i=1}^{n}\int_{\gamma_i}^{\beta_i}\left(\frac{R}{R_0}\right)^{-4} dz + \sum_{i=1}^{n}\int_{\beta_i}^{\gamma_{i+1}} dz + \int_{\beta_n}^{l} dz\right] \qquad (24)$$

Let $G_1 = \sum_{i=1}^{n}\int_{\gamma_i}^{\beta_i}\left(\frac{R}{R_0}\right)^{-1} dz$ and $G_2 = \sum_{i=1}^{n}\int_{\gamma_i}^{\beta_i}\left(\frac{R}{R_0}\right)^{-4} dz$ (25)

$$\lambda = -(h_1 + h_2)\sum_{i=1}^{n+1} di - (h_1 G_1 + h_2 G_2) \qquad (26)$$

In the absence of any constrictions

$$\lambda_f = -(h_1 + h_2)l \qquad (27)$$

Resistance to flow is given by

$$\bar{\lambda} = \frac{\lambda}{\lambda_f} \qquad (28)$$

$$\lambda = \frac{\sum_{j=1}^{n+1} di}{l} + \frac{h_1 G_1 + h_2 G_2}{(h_1 + h_2)l} \qquad (29)$$

Substituting for λ in (1),

$$a_i = 1 - \frac{\delta_i}{2R_0}, \qquad b_i = \frac{\delta_i}{2R_0}, \ \theta = \pi - \frac{2\pi}{l_i}\left(z - \gamma_i - \frac{l_i}{2}\right)$$

$$\frac{R}{R_0} = 1 - b_i(1 - \cos\theta),$$

$z = \gamma_i$ implies that $\theta = 2\pi$, $z = \beta_i$ implies that $\theta = 0$

$$G_1 = \sum_{i=1}^{n}\frac{l_i}{2\pi}\int_0^{2\pi}\frac{1}{a_i + b_i\cos\theta}d\theta = \sum_{i=1}^{n}\frac{l_i}{a_i^2 - b_i^2} \qquad (26)$$

$$G_2 = \sum_{i=1}^{n}\frac{l_i}{2\pi}\int_0^{2\pi}\frac{1}{(a_i - b_i\cos\theta)^4}d\theta = \sum_{i=1}^{n}\frac{a_i(a_i^2 + {}^3/_2 b_i^2)l_i}{(a_i^2 - b_i^2)^{7/2}} \qquad (27)$$

IV. CONCLUSION

Stenosis is a serious cardiovascular disease. The growth of stenosis affects the flow of blood in the arteries and can lead to serious circulatory disorders. Stenosis are formed by the accumulation of fats/ lipids on the inner wall of the arteries and when developed can cause several diseases like blood pressure, atherosclerosis, heart attack and brain hemorrhage.

This research is on going and it is hoped that when this work is concluded, it will be useful to medical scientist in the future diagnosis, understanding and treatment of cardiovascular diseases.

REFERENCE

[1] Sanjeev K. and Chandrashekhar D., Blood flow resistance for a small artery with the effect of multiple stenoses and post stenotic dilatation, *International Journal of Engineering Sciences and Emerging Technologies*, *6(1),* 2013, 57-64.

[2] Sapna R.S, A mathematical model for the analysis of blood flow through diseased blood vessels under the influence of porous parameter, *Journal of Biosciences and Technology*, *4(6),* 2013, 534-541.

[3] Varun and varshney, Pulsatile flow of blood through astenosed artery with body acceleration''. *Ultra Scientist,26 (1),*19-28 (2014)

[4] AmitBhatnaga and R.K Shrivastar , Analysis of MHD flow of blood through amultiple stenosed artery in the presence of slip velocity''. *International Journal of Innovative Research in Advanced Engineering, 1(10),* 2014, 249-256.

[5] Nidhi and Parimar, Mathematical model of blood flow through a tapered artery with mild stenosis and hematocrit, *Journal of Modern Mathematics and Statistics 4(1),* 2010, 38-43.

[6] Hajeet K., Chandel R.S., Sanjeev K. and Sanjeet K., A non-newtonian arterial blood flow model through multiple stenosis, *International Journal of Latest Research in Science and Technology, 3(3),* 2014, 116-121.

[7] Singh A.K. and Singh D.P., Effect of hematocrit on wall shear stress for blood flow through tapered artery, *Applied Bionics and Biomechanics 10,* 2013, 135-138.

[8] Singh A.K., Effect of shape parameter and length of stenoses on blood flow through improved generalized artery with multiple stenosis, *Advances in Applied Mathematical Biosciences, 3(1),* 2012, 41-48.

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http://www.mdcjournals.org/ijsar-jmas...html

[9] Sapna R.S., An innovative solution for the problem of blood flow through stenosed artery using generalized Bingham plastic fluid model, *International Journal of Research in Applied Natural and Social Sciences, 1(3),* 2013, 97-104.

[10] Shailesh M., Siddiqui S.U. and Amit M., Blood flow through a composite stenosis in an artery with permeable wall, *International Journal on Application and Applied Mathematics, 6(11),* 2011, 1798-1813.

[11] Sahu M.K., Sharma S.K. and Agrawal A.K., Study of arterial blood flow in stenosed vessel using non-Newtonian couple stress fluid model, *International Journal of Dynamics of Fluids, 6(2),* 2010, 248-257.

[12] Chakravarty S. and Mandal P.K., Mathematical modeling of blood flow through an overlapping arterial stenosis, *Mathematical Computational Modeling, 19(1),* 1994, 59-70.

[13] Mukhopadhyay S. and Layek G.C., Numerical modeling of a stenosed artery using mathematical model of variable shape, *International Journal on Application and Applied Mathematics, 3(2),* 2008, 308-328.